

# ON BLOCKS STABLY EQUIVALENT TO A QUANTUM COMPLETE INTERSECTION OF DIMENSION 9 IN CHARACTERISTIC 3 AND A CASE OF THE ABELIAN DEFECT GROUP CONJECTURE

RADHA KESSAR

**ABSTRACT.** Using a stable equivalence due to Rouquier, we prove that Broué's abelian defect group conjecture holds for 3-blocks of defect 2 whose Brauer correspondent has a unique isomorphism class of simple modules. The proof makes use of the fact, also due to Rouquier, that a stable equivalence of Morita type between self-injective algebras induces an isomorphism between the connected components of the outer automorphism groups of the algebras.

## 1. INTRODUCTION

The abelian defect group conjecture of Broué [1, Question 6.2], [9, Conjecture, page 132] states that any  $p$ -block ( $p$  a prime number) of a finite group  $G$  whose defect groups are abelian is derived equivalent to its Brauer correspondent. The conjecture is known to hold when the defect groups in question are cyclic [13], Klein 4-groups [14], and when the concerned blocks are nilpotent [11]. The conjecture has also been proved in several situations under additional global assumptions, that is, assumptions on  $G$ . In this paper, we prove the following result.

**Theorem 1.1.** *Let  $k$  be an algebraically closed field of characteristic 3. Let  $G$  be a finite group,  $A$  a block of  $kG$ ,  $P$  a defect group of  $A$  and  $C$  the Brauer correspondent of  $A$  in  $kN_G(P)$ . Suppose that  $P \cong C_3 \times C_3$  and that  $C$  is a non-nilpotent block with a unique isomorphism class of simple modules. Then  $A$  and  $C$  are Morita equivalent as  $k$ -algebras.*

The above theorem is the first of its kind in that it proves a case of the abelian defect group conjecture for all blocks with a prescribed local structure which is of wild representation type and which is not obviously related to the nilpotent case. We point out that we do not know if the equivalence constructed for the proof of Theorem 1.1 is splendid or if it can be lifted to a complete local ring in characteristic zero.

The proof of Theorem 1.1 fits in with the inductive approach to the abelian defect group conjecture outlined by Rouquier in [15] : the starting point is the gluing result of [15], which yields a stable equivalence of Morita type between the blocks  $A$  and  $C$ . Moreover, given the assumptions on  $C$  and Külshammer's structure results on blocks with normal defect groups in [7], it is known that  $C$  is Morita equivalent to the quantum complete intersection algebra  $k\langle X, Y \rangle / \langle X^3, Y^3, XY + YX \rangle$ . Finally, by [12], it is known that  $A$  and  $C$  have isomorphic centers and that  $A$  also has a unique isomorphism class of simple modules. We obtain Theorem 1.1 as a consequence of the following result.

**Theorem 1.2.** *Let  $k$  be an algebraically closed field of characteristic 3. Let*

$$B = k\langle X, Y \rangle / \langle X^3, Y^3, XY + YX \rangle$$

and let  $A$  be a local, symmetric  $k$ -algebra. Suppose that  $\dim_k(A) = 9$ ,  $Z(A) \cong Z(B)$  and that there is a stable equivalence of Morita type between  $A$  and  $B$ . Then  $A$  is isomorphic to  $B$ .

The proof of Theorem 1.2 has two main steps. First, we use arguments in the style of [8] to describe 9-dimensional local symmetric algebras whose center is isomorphic to that of  $B$  - there are infinitely many isomorphism classes of these. In the second step, we show that amongst the algebras obtained in step 1, the only ones which are stably equivalent (à la Morita) to  $B$  are isomorphic to  $B$ . The crucial ingredient here is the fact that a stable equivalence of Morita type between self-injective algebras preserves the connected component of the outer automorphism group of the algebras ([16]).

**Remarks 1.3.** (i) By work of Kiyota in [5], with the notation of Theorem 1.1, it is known that  $A$  has a unique isomorphism class of simple module if and only if  $C$  has a unique isomorphism class of simple modules. Thus Theorem 1.1 remains true if one replaces the hypothesis that  $C$  has a unique isomorphism class of simple modules by the hypothesis that either  $A$  or  $C$  has a unique isomorphism class of simple modules.

(ii) The proof of Theorem 1.2 is very computational; however it seems possible that the methods will find application in other situations, for instance, for  $p$ -blocks of defect 2 whose Brauer correspondent has one isomorphism class of simple modules for other primes  $p$  (see [2], [4]).

## 2. BACKGROUND RESULTS ON ALGEBRAS AND THEIR AUTOMORPHISM GROUPS

Let  $k$  be an algebraically closed field. All  $k$ -algebras will be assumed to be finite dimensional vector spaces over  $k$ . Recall that a  $k$ -algebra  $C$  is symmetric if there exists a  $k$ -linear function  $s : C \rightarrow k$  such that  $s(xy) = s(yx)$  for all  $x, y \in C$  and no non-zero left ideal of  $C$  is contained in the kernel of  $s$ . Also, recall that since  $k$  is algebraically closed,  $C$  is local if and only if  $\dim_k(C/J(C)) = 1$ .

The following gathers some well known properties of local symmetric algebras; we refer to [6] for proofs.

**Lemma 2.1.** *Suppose that  $C$  is a local symmetric  $k$ -algebra. Then*

- (i)  $\dim_k \text{Soc}(C) = 1$ .
- (ii)  $\text{Soc}(C) \subseteq \text{Soc}(Z(C))$ .
- (iii)  $\text{Soc}(C) \cap [C, C] = 0$ .
- (iv)  $\dim_k(C) = \dim_k(Z(C)) + \dim_k[C, C]$ .
- (v)  $Z(C)$  is local and  $Z(C) \cap J(C) = J(Z(C))$ .
- (vi) If  $n$  is the least natural number such that  $J^{n+1}(C) = 0$ , then  $\text{Soc}(C) = J^n(C)$ .

We will also use the following results from [8].

**Lemma 2.2.** [8, Lemma E] *Let  $C$  be a  $k$ -algebra and let  $I$  be a two-sided ideal of  $C$ . Let  $m, n$  be natural numbers with  $m \leq n$ . Suppose that*

$$I^n = k - \text{span of } \{x_{i_1} \cdots x_{i_n} : i = 1, \dots, d\} + I^{n+1}$$

*with elements  $x_{ij} \in I$ . Then,*

$$I^{n+m} = k - \text{span of } \{x_{j_1} \cdots x_{j_m} x_{i_1} \cdots x_{i_n} : i, j = 1, \dots, d\} + I^{n+m+1}.$$

**Lemma 2.3.** [8, Lemma G] *Let  $C$  be a local symmetric  $k$ -algebra. If  $n$  is a natural number such that  $\dim_k J^n(C)/J^{n+1}(C) = 1$ , then  $J^{n-1}(C) \subseteq Z(C)$ .*

For a finite dimensional  $k$ -algebra  $C$ , let  $\text{Aut}(C)$  be the group of automorphisms of  $C$ , viewed as a (linear) algebraic group. Clearly,  $\text{Aut}(C)$  is a subgroup of the algebraic group  $\text{GL}(C)$  of the automorphisms of the  $k$ -vector space  $C$ . For any  $\text{Aut}(C)$ -invariant subspaces  $U, V$  of  $C$  with  $V \subseteq U$ , let  $f_{U,V} : \text{Aut}(C) \rightarrow \text{GL}(U/V)$  denote the map defined by  $f_{U,V}(\varphi)(u + V) = \varphi(u) + V$ ,  $u \in U$ . Any power  $J^i(C)$  of the radical of  $C$  is  $\text{Aut}(C)$ -stable, hence  $\text{Aut}(C)$  is contained in the parabolic subgroup of  $\text{GL}(C)$  of elements which stabilize the flag  $C \supset J(C) \supset J^2(C) \supset \dots$ . For each  $i \geq 0$ , let  $f_i$  denote the map  $f_{J^i(C), J^{i+1}(C)}$ .

Let  $\text{Inn}(C)$  denote the closed connected normal subgroup of  $\text{Aut}(C)$  consisting of inner automorphisms of  $C$  and let  $\text{Out}(C)$  be the quotient group  $\text{Aut}(C)/\text{Inn}(C)$ . Let  $\text{Aut}^0(C)$  denote the connected component of the identity of  $\text{Aut}(C)$  and let  $\text{Out}^0(C) = \text{Aut}^0(C)/\text{Inn}(C)$  denote the connected component of  $\text{Out}(C)$ .

The following easy results will come in handy for calculating automorphism groups. Recall that for a connected linear algebraic group  $\mathbf{G}$  over  $k$ , the unipotent radical  $R_u(\mathbf{G})$  of  $\mathbf{G}$  is the largest closed connected normal unipotent subgroup of  $\mathbf{G}$ .

**Proposition 2.4.** *Suppose that  $C$  is a local algebra. Then,*

- (i)  $\text{Ker}(f_1)$  is a unipotent subgroup of  $\text{Aut}(C)$ .
- (ii)  $\text{Inn}(A) \leq \text{Ker}(f_1)$ .
- (iii) *There exists a bijective morphism of algebraic groups*

$$\hat{f} : \text{Out}^0(C)/R_u(\text{Out}^0(C)) \rightarrow f_1(\text{Aut}(C))^0/R_u(f_1(\text{Aut}(C))^0).$$

The proof of the above is a consequence of the following standard structural result.

**Lemma 2.5.** (i) *Let  $\mathbf{G}$  be a connected linear algebraic group  $\mathbf{G}$  over  $k$ . Then  $R_u(\mathbf{G})$  contains every normal unipotent subgroup of  $\mathbf{G}$ .*

(ii) *Let  $\varphi : \mathbf{G} \rightarrow \mathbf{G}'$  be an epimorphism of connected algebraic groups such that  $\text{Ker}(\varphi)$  is a unipotent subgroup of  $\mathbf{G}$ . Then there is a bijective morphism*

$$\hat{\varphi} : \mathbf{G}/R_u(\mathbf{G}) \rightarrow \mathbf{G}'/R_u(\mathbf{G}')$$

*such that  $\hat{\varphi}(gR_u(\mathbf{G})) = \varphi(g)R_u(\mathbf{G}')$  for all  $g \in \mathbf{G}$ .*

*Proof.* (i) This follows by [3, Theorem 30.4 (b)] and [3, Exercise 26.14].

(ii) Set  $K = \text{Ker}(\varphi)$ . By (i),  $K \leq R_u(\mathbf{G})$  and by [3, Corollary 21.3F, Theorem 30.4 (b), Exercise 26.14],  $\varphi(R_u(\bar{G})) = R_u(\mathbf{G}')$ . The result follows.  $\square$

**Proof of Proposition 2.4.** (i) Since  $C$  is local,  $\text{Aut}(C) = \text{Ker}(f_0)$ . On the other hand,  $\text{Ker}(f_1) \leq \text{Ker}(f_i)$ , for all  $i \geq 1$ . Hence, if  $\varphi \in \text{Ker}(f_1)$ , then  $\varphi$  induces the identity on  $J^i(C)/J^{i+1}(C)$  for all  $i \geq 0$ . This proves (i).

(ii) Let  $x$  be an invertible element of  $C$ . Since  $C$  is local,  $x = \lambda(1+y)$  for some  $0 \neq \lambda \in k$  and some  $y \in J$ . So  $x^{-1} = \lambda^{-1}(1+y')$ ,  $y' \in J$  and conjugation by  $x$  induces the identity on  $J(C)/J^2(C)$ .

(iii) By [3, Proposition 7.4B],  $f_1(\text{Aut}^0(C)) = f_1(\text{Aut}(C))^0$ . So, by (ii), the restriction of  $f_1$  to  $\text{Aut}^0(C)$  yields a surjective morphism

$$\bar{f}_1 : \text{Out}^0(C) \rightarrow f_1(\text{Aut}(C))^0.$$

By (i), the kernel of  $f_1$  is a unipotent subgroup of  $\text{Aut}(C)$ , hence the kernel of  $\bar{f}_1$  is a unipotent subgroup of  $\text{Out}^0(C)$ . So (iii) follows from Lemma 2.5 applied with  $\varphi$  equal to  $\bar{f}_1$ .  $\square$

For  $u, v$  elements of a ring,  $[u, v]$  denotes the commutator  $uv - vu$  of  $u$  and  $v$ .

**Lemma 2.6.** *Let  $C$  be a  $k$ -algebra such that  $J^3(C) \subseteq Z(C)$ . For  $u \in J(C)$ , let  $\varphi_u : C \rightarrow C$  be the automorphism defined by  $\varphi_u(v) = (1 - u)v(1 - u)^{-1}$ ,  $v \in C$ . Then, for any  $u, v \in J(C)$ ,  $\varphi_u(v) = v + [v, u] + [vu, u]$ .*

*Proof.* Let  $u, v \in J(C)$  and suppose that  $u^n = 0$ . Then,  $(1 - u)^{-1} = 1 + u + \cdots + u^{n-1}$ , and for any  $v \in J(C)$ ,

$$\varphi_u(v) = v + [v, u] + [vu, u] + \cdots + [vu^{n-1}, u].$$

Since  $J^3(C) \subseteq Z(C)$ , the result follows.  $\square$

### 3. PRELIMINARY CALCULATIONS

For the rest of the paper, unless stated otherwise,  $k$  will denote an algebraically closed field of characteristic  $p$  and  $B = k\langle X, Y \rangle / \langle X^3, Y^3, XY + YX \rangle$  be as in Theorem 1.2. Throughout this section  $A$  will denote a  $k$ -algebra satisfying the following:

**Hypothesis 3.1.**  $A$  is symmetric, local,  $\dim_k(A) = 9$ , and  $Z(A) \cong Z(B)$  as  $k$ -algebras.

We fix a symmetrizing form  $s : A \rightarrow k$  on  $A$ . For a subset  $U$  of  $A$  denote by  $U^\perp$  the subset of  $A$  consisting of elements orthogonal to  $U$  under the bilinear form associated to  $s$ , that is  $U^\perp$  is the set of elements  $a' \in A$  such that  $s(aa') = 0$  for all  $a \in U$ . For each  $i \geq 0$ , let  $J_i = J^i(A)$ ,  $Z = Z(A)$ ,  $Z_i := Z(A) \cap J^i(A)$ .

**Lemma 3.2.** *Suppose that  $A$  satisfies Hypothesis 3.1. Then,  $Z$  has a  $k$ -basis  $\{1, z_i, 1 \leq i \leq 5\}$  such that*

$$z_1 z_2 = z_2 z_1 = z_5, \quad z_i z_j = 0 \quad \text{for all } i, j \text{ s.t. } \{i, j\} \neq \{1, 2\}$$

*and  $\{z_i, 1 \leq i \leq 5\}$  is a  $k$ -basis of  $Z_1$ . Further,  $\{z_i, 3 \leq i \leq 5\}$  is a  $k$ -basis of  $\text{Soc}(Z(A))$  and  $\dim_k[A, A] = 3$ .*

*Proof.* Since  $Z(A) \cong Z(B)$ , in order to prove the first statement, it suffices to prove an analogous statement for  $Z(B)$ . Set

$$z'_1 := X^2, z'_2 := Y^2, z'_3 := XY^2, z'_4 := YX^2, z'_5 := X^2Y^2 \in Z(B).$$

Then  $\{1, z'_i, i \leq 5\}$  is a basis of  $Z(B)$  and  $z'_1 z'_2 = z'_2 z'_1 = z'_5$  and all other pairwise products are 0. Setting  $z_i$  to be the image of  $z'_i$  under some isomorphism  $Z(A) \cong Z(B)$  yields the first statement. The elements  $z_i$ ,  $1 \leq i \leq 5$  are all nilpotent and central in  $A$ , hence  $z_i \in Z_1$  for  $1 \leq i \leq 5$ . This proves the second assertion. The assertion on  $\text{Soc}(Z(A))$  is immediate from the multiplication of the  $z'_i$  and the last assertion follows by Lemma 2.1.  $\square$

**Lemma 3.3.** *Suppose that  $A$  satisfies Hypothesis 3.1. For  $x \in J$  and  $i \in \mathbb{N}$ , denote by  $C_{J_i}(x)$  the subring of  $J_2$  of elements of  $J_2$  which commute with  $x$ .*

- (i)  $J_2 \not\subseteq Z_2$ .
- (ii)  $J_3 \subseteq Z_3$ .

(iii)  $\dim_k(J_1/(Z_1 + J_2)) = 2$  and  $\dim_k(J_2/Z_2) = 1$ .

(iv) There exists a basis

$$\{x + Z_1 + J_2, y + Z_1 + J_2\}$$

of  $J_1/(Z_1 + J_2)$  such that

$$C_{J_2}(x) = C_{J_2}(y) = Z_2.$$

(v)  $Z_1$  is an ideal of  $A$ .

(vi) For any  $u, v \in J$ ,  $uv + vu \in Z_2$ .

(vii)  $Z_1^2 = \text{Soc}(A)$ .

(viii)  $[A, A] \not\subseteq Z_1$  and  $\text{Soc}(Z(A)) = [A, A] \cap Z_1 \oplus \text{Soc}(A)$ .

(ix)  $J_5 = 0$ ,  $J_4 = Z_1^2 = \text{Soc}(A)$ .

*Proof.* (i) Note that since  $A$  is local, any subspace of  $J_1$  which contains  $J_2$  is an ideal of  $A$ , and further since  $[A, A] \subseteq J_2$ , any such subspace contains  $[A, A]$ .

Suppose if possible that  $J_2 \subseteq Z_2 \subseteq Z_1$ . Then  $Z_1$  and hence  $Z_1^2$  is an ideal of  $A$ . In particular,  $Z_1^2 \cap \text{Soc}(A) \neq 0$ . By Lemma 2.1,  $\text{Soc}(A)$  is 1-dimensional and by Lemma 3.2,  $Z_1^2$  is 1-dimensional. Hence,  $Z_1^2 = \text{Soc}(A)$ , whence by Lemma 2.1,  $[A, A] \cap Z_1^2 = 0$ . On the other hand, since  $[A, A] \subseteq Z_1$ , and since  $za = az$  for any  $a \in A$ ,  $z \in Z_1$ ,

$$[A, A]Z_1 \subseteq [A, A] \cap Z_1^2.$$

Thus,  $[A, A]Z_1 = 0$ , that is  $[A, A] \subseteq \text{Soc}(Z(A))$ . But

$$\dim_K([A, A]) = 3 = \dim_k(\text{Soc}(Z(A))),$$

hence  $[A, A] = \text{Soc}(Z(A))$ . This is a contradiction as by Lemma 2.1 (ii),  $\text{Soc}(A) \subseteq \text{Soc}(Z(A))$  whereas by Lemma 2.1 (iii)  $[A, A] \cap \text{Soc}(A) = 0$ .

(ii) By (i) and Lemma 2.3,  $\dim_k(J_3/J_4) \geq 2$ ,  $\dim_k(J_2/J_3) \geq 2$  and  $\dim_k(J_1/J_2) \geq 2$ . So,  $\dim_k(J_4) \leq 2$ . By Lemma 2.1 (i),(vi), it follows that  $\dim_k(J_4/J_5) \leq 1$ . Hence by Lemma 2.3,  $J_3 \subseteq Z$ .

(iii) Let  $c$  be the codimension of  $J_2 + Z_1$  in  $J_1$ . Let  $\mathcal{A}$  be a basis of a complement of  $J^2(A) + Z_1$  in  $J(A)$  and let  $\mathcal{A}'$  be a (possibly empty) basis of a complement of  $J^2(A)$  in  $J^2(A) + Z_1$  consisting of elements of  $Z_1$ . Then  $\{1\} \cup \mathcal{A} \cup \mathcal{A}'$  is a generating set for  $A$  as an algebra. If the elements of  $\mathcal{A}$  pairwise commute, then  $\{1\} \cup \mathcal{A} \cup \mathcal{A}'$  is a set of pairwise commuting generators of  $A$ , an impossibility as  $A$  is not commutative. So  $c \geq 2$ . On the other hand, the codimension of  $Z_1$  in  $J_1$  is 3, so  $c \leq 3$ . Suppose if possible that  $c = 3$ . Then

$$\dim_k(J_2 + Z_1) = 5 = \dim_k(Z_1),$$

hence  $J_2 \subseteq Z_1$ , a contradiction to (i). Thus,  $c = 2$ . The second assertion is immediate from the first and (i).

(iv) Let  $\mathcal{A} = \{x, y\}$  and  $\mathcal{A}'$  be as in (ii). Since  $\{1\} \cup \mathcal{A} \cup \mathcal{A}'$  is a generating set for  $A$  as an algebra,

$$Z_2 = C_{J_2}(x) \cap C_{J_2}(y).$$

As  $\dim_k Z_1 = 5$  and  $x \in C_{J_1}(x)$ ,  $\dim_k C_{J_1}(x) \geq 6$ . Similarly,  $\dim_k C_{J_1}(y) \geq 6$ . On the other hand, since  $\dim_k J_1 = 8$ , and  $\dim_k Z_1 = 5$ , at least one of  $\dim_k C_{J_1}(x)$  and  $\dim_k C_{J_1}(y)$  is at most 6. Suppose that  $\dim_k C_{J_1}(x) = 6$  and  $\dim_k C_{J_1}(y) = 7$ . Then by the same argument applied to  $\{x + y, y\}$ , it follows that  $\dim_k C_{J_1}(x + y) = 6$ . Hence replacing  $y$  with  $x + y$ , we may assume that

$$\dim_k C_{J_1}(x) = 6 = \dim_k C_{J_1}(y).$$

Thus,  $C_{J_1}(x)$  is the  $k$ -span of  $\{x, Z_1\}$  and  $C_{J_2}(y)$  is the  $k$ -span of  $\{y, Z_1\}$ . Consequently,  $C_{J_2}(x) \subseteq Z_2$  and  $C_{J_2}(y) \subseteq Z_2$ .

From now on let  $\mathcal{A} = \{x, y\} \subseteq J_1$  be such that  $\{x + Z_1 + J_2, y + Z_1 + J_2\}$  is a basis of  $J_1/(Z_1 + J_2)$  and  $C_{J_2}(x) = C_{J_2}(y) = Z$ . Let  $\mathcal{A}'$  be a basis of a complement of  $J_2$  in  $J_2 + Z_1$  consisting of elements of  $Z_1$ .

(v) Since  $A$  is local, and since  $\mathcal{A}' \subseteq Z_1$ , it suffices to prove that  $xZ_1 \subseteq Z_2$  and  $yZ_1 \subseteq Z_2$ . So, let  $z \in Z_1$ . Then,  $xz \in C_{J_2}(x)$ , so by the above  $xz \in Z_2$ . Similarly,  $yz \in Z_2$ .

(vi) By (ii),  $J_3 \subseteq Z_3$  and by (v),  $Z_1$  is an ideal of  $A$ . Hence it suffices to prove that  $x^2, y^2, xy + yx \in Z_2$ . Since  $x^2 \in C_{J_2}(x)$ ,  $x^2 \in Z_2$ . Similarly,  $y^2 \in Z_2$ . Also,

$$x(xy + yx) = x^2y + xyx = yx^2 + xyx = (xy + yx)x$$

and similarly,

$$y(xy + yx) = (xy + yx)y,$$

so  $xy + yx \in Z_2$ .

(vii) By (v),  $Z_1$  is an ideal of  $A$ . So,  $0 \neq Z_1^2$  is an ideal of  $A$ , and hence  $\text{Soc}(A) \subseteq Z_1^2$ . But since  $Z_1^2$  is one dimensional, it follows that  $\text{Soc}(A) = Z_1^2$ .

(viii) Since  $Z_1([A, A] \cap Z_1) \subseteq [A, A] \cap Z_1^2$  and since by (vii),  $Z_1^2 = \text{Soc}(A)$ , by Lemma 2.1 (iii),

$$([A, A] \cap Z_1)Z_1 = 0.$$

Hence  $[A, A] \cap Z_1 \subseteq \text{Soc}(Z(A))$  and again by Lemma 2.1 (iii),  $[A, A] \cap Z_1 \cap \text{Soc}(A) = 0$ . The result follows since by (iii), the co-dimension of  $[A, A] \cap Z_1$  in  $[A, A]$  is at most 1.

(ix) Let  $x_i \in \mathcal{A} \cup \mathcal{A}'$ ,  $1 \leq i \leq 5$ . If one of the  $x_i$ 's is in  $\mathcal{A}'$ , then by (vi) and (vii)

$$x_1x_2x_3x_4x_5 \in Z_1J_3J_1 \subseteq Z_1^2J_1 = 0.$$

So, in order to show that  $J_5 = 0$ , we may assume that  $x_i \in \mathcal{A}$ ,  $1 \leq i \leq 5$ . First suppose that for some  $i$ ,  $1 \leq i \leq 4$ ,  $x_i = x_{i+1}$ , say  $x_i = x_{i+1} = x$ . By (v) and (vi), for some  $r, s \geq 0$  such that  $r + s = 5$ ,

$$x_1x_2x_3x_4x_5 = x^rx^s \in Z_1^2J_1 = 0.$$

Suppose now that no two consecutive  $x_i$ 's are equal, so

$$x_1x_2x_3x_4x_5 = xyxyx$$

or

$$x_1x_2x_3x_4x_5 = yxyxy.$$

In the former case, by (ii) and (v),

$$x_1x_2x_3x_4x_5 = x^2yxy$$

and we are back in the previous situation. The second case is similar.

Suppose if possible that  $J_4 = 0$ . Then  $J_3 = \text{Soc}(A)$  is 1-dimensional, and by Lemma 2.3,  $J_2 \subseteq Z_1$ , a contradiction to (i).  $\square$

**Lemma 3.4.** *Suppose that  $A$  satisfies Hypothesis 3.1 and let  $\{x + Z_1 + J_2, y + Z_1 + J_2\}$  be a basis of  $J_1/(Z_1 + J_2)$ . Then,  $\{xy + Z_2\}$  is a basis of  $J_2/Z_2$ ,  $\dim_k[A, A] + J_3/J_3 = 1$ ,  $\{[x, y] + J_3\}$  is a basis of  $([A, A] + J_3)/J_3$  and  $\{[x, y], [x, xy], [y, xy]\}$  is a basis of  $[A, A]$ ,*

*Proof.* By Lemma 3.3 (v) and (vi),  $Z_1 J_1 \subseteq Z_2$  and  $x^2, y^2, xy + yx \in Z_2$ , and by Lemma 3.3 (i),  $J_2 \not\subseteq Z_2$ . Hence,  $xy \notin Z_2$  and it follows by Lemma 3.3 (iii) that  $\{xy + Z_2\}$  is a basis of  $J_2/Z_2$ .

By Lemma 3.3 (ii),  $J_3 \subseteq Z$  and by Lemma 3.3 (viii)  $[A, A] \not\subseteq Z$ . So  $[A, A] \not\subseteq J_3$ . On the other hand,  $[A, A] + J_3/J_3$  is spanned by  $[x, y]$ . So,  $\{[x, y] + J_3\}$  is a basis of  $[A, A] + J_3/J_3$  and  $\dim_k([A, A] + J_3/J_3) = 1$ . It follows from this that  $\{x, y, [x, y], Z_1\}$  spans  $J_1$ , and hence that  $\{[x, y], [x, xy], [y, xy]\}$  spans  $[A, A]$ . Since  $\dim_k[A, A] = 3$ ,  $\{[x, y], [x, xy], [y, xy]\}$  is a basis of  $[A, A]$ . □

**Proposition 3.5.** *Suppose that  $A$  satisfies Hypothesis 3.1. Then one of the following holds.*

$$(1) \quad \dim_k(J_1/J_2) = 3, \dim_k(J_2/J_3) = 2, \dim_k(J_3/J_4) = 2, \dim_k(J_4) = 1$$

$$(2) \quad \dim_k(J_1/J_2) = 2, \dim_k(J_2/J_3) = 3, \dim_k(J_3/J_4) = 2, \dim_k(J_4) = 1$$

Moreover,  $Z_1 J_1 \subseteq J_3$ .

*Proof.* As in the proof of Lemma 3.3 (ii),

$$\dim_k(J_3/J_4) \geq 2, \dim_k(J_2/J_3) \geq 2, \dim_k(J_1/J_2) \geq 2.$$

By Lemma 3.3,  $J_4 = \text{Soc}(A)$  is 1-dimensional. Thus, we have the following possibilities:

$$\dim_k(J_1/J_2) = 3, \dim_k(J_2/J_3) = 2, \dim_k(J_3/J_4) = 2, \dim_k(J_4) = 1$$

$$\dim_k(J_1/J_2) = 2, \dim_k(J_2/J_3) = 3, \dim_k(J_3/J_4) = 2, \dim_k(J_4) = 1,$$

$$\dim_k(J_1/J_2) = 2, \dim_k(J_2/J_3) = 2, \dim_k(J_3/J_4) = 3, \dim_k(J_4) = 1.$$

We show that the last is not possible. Suppose, if possible that  $J_1/J_2$  has a basis  $\{x + J_2, y + J_2\}$  and that  $\dim_k(J_2/J_3) = 2$ . Then  $\{xy + J_3, yx + J_3, x^2 + J_3, y^2 + J_3\}$  is a spanning set for  $J_2/J_3$ . By Lemma 3.3 (vi),  $x^2, y^2, xy + yx \in Z_2$  and by Lemma 3.3(i),  $J_2 \not\subseteq Z_2$ . So,  $\{xy + yx + J_3, x^2 + J_3, y^2 + J_3\}$  spans  $Z_2/J_3$ . Suppose first that  $x^2, y^2 \in J_3$ . Since  $Z_2 + J_3$  has codimension 1 in  $J_2$ ,  $xy + yx \notin J_3$ . So, replacing  $x$  with  $x' = x + y$ , we may assume that  $x^2 \notin J_3$  and hence that  $\{x^2 + J_3\}$  is a basis of  $Z_2/J_3$ . Thus, by Lemma 3.4,  $\{xy + J_3, x^2 + J_3\}$  is a basis of  $J_2/J_3$ . But then, by Lemma 2.2,  $\{x^2 y + J_4, x^3 + J_4\}$  is a spanning set of  $J_3/J_4$ , a contradiction. This proves the first assertion.

If  $\dim_k(J_1/J_2) = 2$ , then  $Z_1 \subseteq J_2$ , and the second statement is trivial. Thus, we may assume that  $\dim_k(J_1/J_2) = 3$  and hence by the first assertion that  $\dim_k(J_2/J_3) = 2$ . Let  $\{x + J_2, y + J_2, z + J_2\}$  be a basis of  $J_1/J_2$  with  $z \in Z$  and suppose if possible that  $Z_1 J_1 \not\subseteq J_3$ . Since by Lemma 3.3 (iii),  $Z_2$  has codimension 1 in  $J_2$ ,  $\{x + Z_1 + J_2, y + Z_1 + J_2\}$  is a basis of  $J_1/(Z_1 + J_2)$ . By Lemma 3.3(viii), (ix),  $z^2 \in \text{Soc}(Z(A)) = J_4$ , hence either  $xz \notin J_3$  or  $yz \notin J_3$ , say  $xz \notin J_3$ . By Lemma 3.3(v),  $xz \in Z_2$  and by Lemma 3.4,  $\{xy + Z_2\}$  is a basis of  $J_2/Z_2$ . Thus,  $\{xy + J_3, xz + J_3\}$  is a basis of  $J_2/J_3$ . By Lemma 2.2,  $\{x^2 y + J_4, x^2 z + J_4\}$  spans  $J_3/J_4$ . But by Lemma 3.3(vi),  $x^2 \in Z_1$  whence by Lemma 3.3(vii), (ix)

$$x^2 z \in Z_1^2 = \text{Soc}(A) = J_4.$$

So  $(\dim_k J_3/J_4) \leq 1$ , a contradiction. □

#### 4. THE CASE $\dim_k J_1/J_2 = 3$ .

In this section, we will work under the following hypothesis.

**Hypothesis 4.1.**  $A$  is symmetric, local,  $\dim_k(A) = 9$ ,  $Z(A) \cong Z(B)$  as  $k$ -algebras and  $\dim_k(J_1/J_2) = 3$ .

With the above hypothesis, by Lemma 3.3  $J_1/J_2$  has a basis of the form  $\{x + J_2, y + J_2, z + J_2\}$ , with  $z \in Z_1$ .

**Lemma 4.2.** *Suppose  $A$  satisfies Hypothesis 4.1. There exists a basis  $\{x + J_2, y + J_2, z + J_2\}$  of  $J_2/J_3$  such that  $z \in Z$ ,  $x^2 \notin J_3$ , and  $xy + yx = 0$ .*

*Proof.* Let  $\{x + J_2, y + J_2, z + J_2\}$  be a basis of  $J_2/J_3$  with  $z \in Z$ . By Proposition 3.5,  $xz, zx \in J_3$ , so  $\{x^2 + J_3, y^2 + J_3, xy + J_3, yx + J_3\}$  spans  $J_2/J_3$ . By Lemma 3.3 (vi),  $x^2, y^2, xy + yx \in Z_2$  and by Lemma 3.3 (i),  $J_2 \not\subseteq Z_2$ . Thus,  $\{x^2 + J_3, y^2 + J_3, xy + yx + J_3\}$  spans  $Z_2/J_3$ . If either  $x^2 \notin J_3$  or  $y^2 \notin J_3$ , then interchanging  $x$  and  $y$  if necessary, we have  $x^2 \notin J_3$ . If both  $x^2, y^2 \in J_3$  then  $xy + yx \notin J_3$ . So,

$$(x + y)^2 = x^2 + y^2 + (xy + yx) \notin J_3,$$

and replacing  $x$  with  $x + y$  we may assume that  $x^2 \notin J_3$ . Thus, since  $J_2 \not\subseteq Z_2$  and  $\dim_k(J_2/J_3) = 2$ ,  $\{x^2 + J_3\}$  is a basis of  $Z_2/J_3$ .

Since  $xy + yx \in Z_2$ , there exists an  $\alpha \in k$  such that

$$xy + yx \equiv \alpha x^2 \pmod{J_3}$$

Set  $y' = y - \frac{1}{2}\alpha x$ . Then,

$$xy' + y'x \equiv 0 \pmod{J_3}.$$

Since  $\{x + J_2, y' + J_2, z + J_2\}$  is also a basis of  $J_1/J_2$ , by replacing  $y$  by  $y'$ , we may assume that  $xy + yx \in J_3$ .

Since  $\{x^2 + J_3\}$  is a basis of  $Z_2/J_3$ , by Lemma 3.4,  $\{xy + J_3, x^2 + J_3\}$  is a basis of  $J_2/J_3$ , hence by Lemma 2.2,  $\{x^2y, x^3\}$  is a basis of  $J_3/J_4$ . So,

$$xy + yx \equiv \alpha x^3 + \beta x^2y \pmod{J_4}$$

for some  $\alpha, \beta \in k$ .

Set  $x' = x - \frac{1}{2}\beta x^2$  and  $y' = y - \frac{1}{2}\alpha x^2$ . Then  $\{x' + J_2, y' + J_2, z + J_2\}$  is still a basis of  $J_1/J_2$ ,  $z \in Z$ ,  $x'^2 \notin J_3$  and

$$\begin{aligned} x'y' + y'x' &\equiv (x - \frac{1}{2}\beta x^2)(y - \frac{1}{2}\alpha x^2) + (y - \frac{1}{2}\alpha x^2)(x - \frac{1}{2}\beta x^2) \\ &\equiv xy + yx - \alpha x^3 - \beta x^2y \\ &\equiv 0 \pmod{J_4} \end{aligned}$$

Hence replacing  $x$  with  $x'$  and  $y$  with  $y'$ , we may assume that  $xy + yx \in J_4$ . Arguing as above, again  $\{x^2y, x^3\}$  is a basis of  $J_3/J_4$  and by applying Lemma 2.2 again,  $\{x^3y, x^4\}$  spans  $J_4$ . Since  $\dim_k(J_4) = 1$ , either  $xy + yx = \alpha x^3y$  or  $xy + yx = \alpha x^4$  for some  $\alpha \in k$ . Replacing  $x$  by  $x - \frac{1}{2}\alpha x^3$  in the first case and  $y$  by  $y - \frac{1}{2}\alpha x^3$  in the second case yields the result.  $\square$



**Lemma 4.3.** *Suppose  $A$  satisfies Hypothesis 4.1 and let  $\{x + J_2, y + J_2, z + J_2\}$  be a basis  $J_2/J_3$  such that  $z \in Z$ ,  $x^2 \notin J_3$  and  $xy + yx = 0$ . Then,*

(i) *For any  $u \in Z_1$ ,  $uxy = xyu = 0$ . In particular,*

$$x^3y = yx^3 = xy^3 = yx^3 = 0.$$

(ii)  *$\{xy + J_3, x^2 + J_3\}$  is a basis of  $J_2/J_3$ ,  $\{x^2y + J_4, x^3 + J_4\}$  is a basis of  $J_3/J_4$  and  $\{x^4\}$  is a basis of  $J_4$ .*

(iii)  *$y^2 \notin J_3$ .*

*Proof.* (i) Let  $u \in Z_1$ . Since  $xy = -yx$ , and since by Lemma 3.3 (v),  $uy \in Z$ , we have

$$uxy = -uyx = -x(uy) = -x(yu) = -uxy.$$

Thus,  $uxy = 0$ . The second assertion follows from the first as by Lemma 3.3 (vi),  $x^2, y^2 \in Z_2$ .

(ii) The first assertion was proved in the course of the proof of Lemma 4.2. It follows from this and Lemma 2.2 that  $J_3/J_4$  is spanned by  $\{x^2y + J_4, x^3 + J_4\}$  and  $J_4$  is spanned by  $\{x^3y, x^4\}$ . But by (i),  $x^3y = 0$ . Hence  $\{x^4\}$  is a basis of  $J_4$ .

(iii) By Lemma 3.4,  $\{[x, y], [x, xy], [y, xy]\}$  is a basis of  $[A, A]$ . Since  $xy = -yx$ ,  $[y, [xy]] = 2xy^2$ . If  $y^2 \in J_3$ , then

$$[y, [xy]] = 2xy^2 \in J_4 = \text{Soc}(A),$$

a contradiction. □

**Lemma 4.4.** *Suppose  $A$  satisfies Hypothesis 4.1. Then there exists a basis  $\{x + J_2, y + J_2, z + J_2\}$  of  $J_2/J_3$  such that  $z \in Z$ ,  $\{x^2 + J_3\}$  is a basis of  $Z_2/J_3$ ,  $xy + yx = 0$  and such that the following holds.*

(i)  *$zx = xz = 0$ .*

(ii)  *$z^2 = x^4$ .*

(iii)  *$zy = yz = 0$ .*

(iv)  *$y^2 = x^2 + \alpha x^3 + \beta x^2y$  for some  $\alpha, \beta \in k$ .*

*Proof.* Let  $\{x + J_2, y + J_2, z + J_2\}$  be a basis  $J_2/J_3$  such that  $z \in Z$ ,  $\{x^2 + J_3\}$  is a basis of  $Z_2/J_3$  and  $xy + yx = 0$ . Such a basis exists by Lemma 4.2.

By Lemma 3.3,  $y^2 \in Z_2$ , and by Lemma 4.3,  $y^2 \notin J_3$ , so  $y^2 \cong \lambda x^2 \pmod{J_3}$ , for some  $0 \neq \lambda \in k$ . Replacing  $y$  by  $\sqrt{\lambda}^{-1}y$  does not affect the relation  $xy + yx = 0$ , hence we may assume that  $y^2 \cong x^2 \pmod{J_3}$ . In particular,  $x^2y^2 = x^4$ .

By Proposition 3.5,  $zx \in J_3$ . So, by Lemma 4.3,  $zx = \alpha x^3 + \beta x^2y + \gamma x^4$  with  $\alpha, \beta, \gamma \in k$ . By Lemma 4.3 (i), and by (i),

$$0 = zxy = \alpha x^3y + \beta x^2y^2 = \beta x^2y^2 = \beta x^4.$$

So,  $\beta = 0$ . Since  $z' := z - \alpha x^2 - \gamma x^3$  is an element of  $Z$  and  $z' \equiv z \pmod{J_2}$ , replacing  $z$  by  $z'$ , we may assume that  $zx = xz = 0$  and (i) holds.

Since  $\{x^2, z, x^2y, x^3, x^4\} \subseteq Z_1$  and  $\dim_k(Z_1) = 5$ ,  $\{x^2, z, x^2y, x^3, x^4\}$  is a basis of  $Z_1$  by Lemma 4.3. Now  $\{x^2y, x^3, x^4\} \subseteq \text{Soc}(Z(A))$  and  $\dim_k(\text{Soc}(Z(A))) = 3$ , so  $z \notin \text{Soc}(Z(A))$ . Since  $zx^2 = 0$ , this means that  $z^2 \neq 0$ . On the other hand, by Lemma 3.3 (vii),  $z^2 \in \text{Soc}(Z(A)) = \text{Soc}(A)$ . Thus,  $z^2 = \delta x^4$ , with  $0 \neq \delta \in k$ . Replacing  $z$  with a constant multiple does not affect any of the already established properties. Hence, we may assume that  $z^2 = x^4$  and (ii) holds.

We now show (iii). Again by Proposition 3.5,  $zy \in J_3$ , so by Lemma 4.3,

$$zy = \alpha x^3 + \beta x^2 y + \delta x^4,$$

with  $\alpha, \beta, \delta \in k$ . By Lemma 4.3,

$$0 = zyx = \alpha x^4,$$

whence  $\alpha = 0$ . Since  $zJ \subseteq J_3$ ,  $zJ_3 = 0$ , and we have shown above that  $y^2 \cong x^2 \pmod{J_3}$ . Hence by (i),

$$0 = zx^2 = zy^2 = \beta x^2 y^2 = \beta x^4.$$

But then  $\beta = 0$  and  $zy = \delta x^4$ . Set  $y' = y - \delta z$ . Then by (i)

$$y'x = yx = -xy = -xy',$$

by (i), (ii) and Proposition 3.5,

$$y'^2 \equiv y^2 + \delta^2 z^2 - 2\delta yz \equiv y^2 \equiv x^2 \pmod{J_3},$$

and by (ii),

$$y'z = yz - \delta z^2 = 0.$$

Replacing  $y$  with  $y'$  gives (ii).

Since  $y^2 \equiv x^2 \pmod{J_3}$ ,  $y^2 = x^2 + \alpha x^3 + \beta x^2 y + \gamma x^4$  for some  $\alpha, \beta, \gamma \in k$ .

Set  $y' = y + \sqrt{\gamma}xy$ . Then  $y'x + xy' = 0$ ,  $y'z = zy' = 0$  and by Lemma 4.3,

$$y'^2 = y^2 - \gamma x^2 y^2 = x^2 + \alpha x^3 + \beta x^2 y = x^2 + \alpha x^3 + \beta x^2 y'.$$

So, replacing  $y$  by  $y'$  yields (iv). □

**Proposition 4.5.** *Suppose  $A$  satisfies Hypothesis 4.1. Then there exists a basis  $\{x + J_2, y + J_2, z + J_2\}$  of  $J_1/J_2$  such that  $\{xy + J_3, yx + J_3\}$  is a basis of  $J_2/J_3$ ,  $\{xyx + J_3, yxy + J_3\}$  is a basis of  $J_3/J_4$ ,  $\{xyxy\}$  is a basis of  $J_4$  and such that the following relations hold.*

(i)  $zx = xz = zy = yz = 0$ .

(ii)  $z^2 = xyxy$ .

(iii)  $y^2 = x^2 = \alpha xyx + \beta yxy$ ,  $\alpha, \beta \in k$ .

(iv)  $xyxy = yxyx$ ,

(v) For any  $u_i \in \{x, y, z\}$ ,  $1 \leq i \leq 5$ ,  $u_1 u_2 u_3 u_4 u_5 = 0$ .

Moreover, the above is a complete set of generators and relations for the algebra  $A$ , that is  $A \cong k\langle x, y, z \rangle / I$ , where  $I$  is the ideal of  $k\langle x, y, z \rangle$ , generated by:

$\{zx, xz, zy, yz, z^2 - xyxy, y^2 - x^2, y^2 - \alpha xyx + \beta yxy\} \cup \{u_1 u_2 u_3 u_4 u_5, u_i \in \{x, y, z\}, 1 \leq i \leq 5\}$ ,  $\alpha, \beta \in k$ .

*Proof.* Let  $\{x + J_2, y + J_2, z + J_2\}$  be a basis of  $J_1/J_2$  satisfying the conditions of Lemma 4.4. Let  $i$  be a primitive 4-th root of unity in  $k$  and set  $x' = x + iy$ ,  $y' = x - iy$ ,  $z' = z$ . Then  $x'^2 = y'^2 \in J_3$  and  $x'z' = z'x' = z'y' = y'z' = 0$ . A monomial of two terms in  $x', y', z'$  which involves any of  $x'^2, y'^2$  or  $z'$  is in  $J_3$ , hence  $\{x'y' + J_3, y'x' + J_3\}$  spans  $J_2/J_3$  and is therefore a basis of  $J_2/J_3$ . Similarly, a monomial of three terms in  $x', y', z'$  which involves any of  $x'^2, y'^2$  or  $z'$  is in  $J_4$ , hence  $\{x'y'x' + J_4, y'x'y' + J_4\}$  is a basis of  $J_3/J_4$  and  $J_4$  is spanned by  $x'y'x'y' = y'x'y'x'$ . Write

$$x'^2 = y'^2 = \alpha' x'y'x' + \beta' y'x'z' + \delta x'y'x'y'.$$

Replacing  $x'$  by  $x' - \frac{\delta}{2}y'x'y'$  and  $y'$  by  $y' - \frac{\delta}{2}x'y'x'$  yields

$$x'^2 = y'^2 = \alpha' x'y'x' + \beta' y'x'y'.$$

Since  $z'^2$  is a non-zero element of  $J_4$ , replacing  $z'$  by  $\delta z'$  for a suitable  $\delta$ , we may assume that  $z'^2 = x'y'x'y'$ . Thus,  $\{x', y', z'\}$  satisfy the relations (i) -(v). The final assertion follows easily from this and the fact that  $\dim_k(A) = 9$ .  $\square$

For the sake of completeness we record the following without proof.

**Proposition 4.6.** *For any  $\alpha, \beta \in k$ , the algebra  $k\langle x, y, z \rangle / I$ , where  $I$  is the ideal generated by  $\{zx, xz, zy, yz, z^2 - xyxy, y^2 - x^2, y^2 - \alpha xyx + \beta yxy\} \cup \{u_1 u_2 u_3 u_4 u_5, u_i \in \{x, y\}, 1 \leq i \leq 5\}$  is a local symmetric  $k$ -algebra of dimension 9 and with center isomorphic to  $Z(B)$ .*

## 5. THE CASE $\dim_k(J_1/J_2) = 2$ .

The following lemma divides the case that  $\dim_k(J_1/J_2) = 2$  in two subcases.

**Lemma 5.1.** *Suppose that  $A$  is symmetric, local,  $\dim_k(A) = 9$ ,  $Z(A) \cong Z(B)$  as  $k$ -algebras and  $\dim_k(J_1/J_2) = 2$ . Then there exists a basis  $\{x + J_2, y + J_2\}$  of  $J_2/J_3$  such that either  $xy + yx \in J_3$  or  $y^2 \in J_3$ .*

*Proof.* By Proposition 3.5,  $\dim_k(J_2/J_3) = 3$ , and by Lemma 3.3,

$$\dim_k(Z_2/J_3) = 2, \dim_k(J_2/Z_2) = 2.$$

Let  $\{x + J_2, y + J_2\}$  be a basis of  $J_1/J_2$ . Then,  $\{xy + J_3, yx + J_3, x^2 + J_3, y^2 + J_3\}$  is a spanning set for  $J_2/J_3$ . By Lemma 3.3,  $\{xy + yx, x^2, y^2\} \subseteq Z_2$  and  $J_2 \not\subseteq Z_2$ , so  $\{xy + yx + J_3, x^2 + J_3, y^2 + J_3\}$  spans  $Z_2/J_3$  and some two element subset of  $\{xy + yx + J_3, x^2 + J_3, y^2 + J_3\}$  is a basis of  $Z_2/J_3$ .

Suppose first that  $\{x^2 + J_3, y^2 + J_3\}$  is a basis of  $Z_2/J_3$ . Let  $xy + yx \equiv \lambda x^2 + \mu y^2 \pmod{J_3}$ ,  $\lambda, \mu \in k$ . If  $\lambda = \mu = 0$ , then  $xy + yx \in J_3$ . So, suppose that  $\lambda \neq 0$ . By replacing  $y$  with  $\lambda^{-1}y$ , we may assume that  $\lambda = 1$ . First consider the case that  $\mu \neq 1$ . Set

$$\sigma = -1 + \sqrt{1 - \mu}, \tau = -1 - \sqrt{1 - \mu}, x' = x + \sigma y, y' = x + \tau y.$$

Then since  $\sigma \neq \tau$ ,  $\{x' + J_2, y' + J_2\}$  is a basis of  $J_1/J_2$  and

$$x'y' + y'x' \equiv 0 \pmod{J_3}.$$

So, replacing  $\{x, y\}$  with  $\{x', y'\}$  proves the result. Now consider the case  $\mu = 1$  and set  $y' = y - x$ . Then

$$y'^2 = y^2 + x^2 - (yx + xy) \in J_3.$$

Replacing  $y$  by  $y'$  yields the result.

Now suppose that  $\{xy + yx + J_3, x^2 + J_3\}$  is a basis of  $Z_2/J_3$  and let

$$y^2 \equiv \alpha x^2 + \beta(xy + yx) \pmod{J_3}$$

for  $\alpha, \beta \in k$ . Set  $y' = y - \beta x$ . Then,

$$y'^2 = y^2 + \beta^2 x^2 - \beta(xy + yx) \equiv (\alpha + \beta^2)x^2 \pmod{J_3}.$$

So, replacing  $y$  by  $y'$  we may assume that

$$y^2 \equiv \gamma x^2 \pmod{J_3}, \gamma \in k$$

If  $\gamma = 0$ , the result is proved. Otherwise, replacing  $y$  by  $\gamma^{-1}y$ , we may assume that

$$y^2 \equiv x^2 \pmod{J_3}.$$

Now set  $x' = x + y$  and  $y' = x - y$ . Then  $\{x' + J_2, y' + J_2\}$  is a basis of  $J_1/J_2$  and

$$x'y' + y'x' \in J_3.$$

So, replacing  $\{x + J_2, y + J_2\}$  with  $\{x' + J_2, y' + J_2\}$  yields the result.  $\square$

In view of the above Lemma, for the rest of the section, we will work under one of the following two hypotheses.

**Hypothesis 5.2.**  $A$  is symmetric, local,  $\dim_k(A) = 9$ ,  $Z(A) \cong Z(B)$  as  $k$ -algebras  $\dim_k(J_1/J_2) = 2$  and  $J_1/J_2$  has a basis  $\{x + J_2, y + J_2\}$  with  $xy + yx \in J_3$ .

**Hypothesis 5.3.**  $A$  is symmetric, local,  $\dim_k(A) = 9$ ,  $Z(A) \cong Z(B)$  as  $k$ -algebras,  $\dim_k(J_1/J_2) = 2$  and  $J_1/J_2$  has a basis  $\{x + J_2, y + J_2\}$  such that  $y^2 \in J_3$ .

**Lemma 5.4.** *Suppose that  $A$  satisfies Hypothesis 5.2. Then  $x$  and  $y$  may be chosen such that  $xy + yx = 0$ .*

*Proof.* Since  $xy + yx \in J_3$ ,  $[x, [xy]] \equiv 2x^2y \pmod{J_4}$  and  $[y, [xy]] \equiv -2xy^2 \pmod{J_4}$ . On the other hand, by Lemma 3.4, and the fact that  $[A, A] \cap J_4 = 0$ ,  $\{[x, [xy]] + J_4, [y, [xy]] + J_4\}$  is a basis of  $J_3/J_4$ . Thus,  $\{x^2y + J_4, xy^2 + J_4\}$  is a basis of  $J_3/J_4$ . So,

$$xy + yx \equiv \lambda x^2y + \mu xy^2 \pmod{J_4}.$$

Set  $x' = x - \frac{\lambda}{2}x^2$  and  $y' = y - \frac{\mu}{2}y^2$ . Then  $\{x', y'\}$  is a basis of  $J_1/J_2$  and by Lemma 3.3,

$$x'y' + y'x' \equiv (x - \frac{\lambda}{2}x^2)(y - \frac{\mu}{2}y^2) + (y - \frac{\mu}{2}y^2)(x - \frac{\lambda}{2}x^2) \equiv 0 \pmod{J_4}$$

So, replacing  $\{x, y\}$  with  $\{x', y'\}$ , we may assume that  $xy + yx \equiv 0 \pmod{J_4}$ . By the first part of the argument,  $\{x^2y + J_4, xy^2 + J_4\}$  is a basis of  $J_3/J_4$ . Hence, by Lemma 2.2  $xy + yx = \lambda x^2y^2$  or  $xy + yx = \lambda x^3y$  for some  $\lambda \in k$ . In the first case, replacing  $y$  by  $y' := y - \frac{\lambda}{2}xy^2$  and in the second case replacing  $y$  by  $y' := y - \frac{\lambda}{2}x^2y$  yields the result.  $\square$

**Proposition 5.5.** *Suppose that  $A$  satisfies Hypothesis 5.2. Then there exists a basis  $\{x + J_2, y + J_2\}$  of  $J_1/J_2$  such that  $\{x^2 + J_3, y^2 + J_3, xy + J_3\}$  is a basis of  $J_2/J_3$ ,  $\{x^2y + J_4, xy^2 + J_4\}$  is a basis of  $J_3/J_4$ ,  $\{x^2y^2\}$  is a basis of  $J_4$ , and  $J^5 = \{0\}$  and such that the following relations hold.*

- (i)  $xy + yx = 0$ .
- (ii)  $x^3 = \alpha xy^2 + \beta x^2y^2$  and  $y^3 = \gamma x^2y + \delta x^2y^2$  for some  $\alpha, \beta, \gamma, \delta \in k$ , with  $\alpha, \gamma \in \{0, 1\}$ .
- (iii)  $x^3y = xy^3 = 0$ .
- (iv) For any  $u_i \in \{x, y\}$ ,  $1 \leq i \leq 5$ ,  $u_1u_2u_3u_4u_5 = 0$ .

The above is a complete set of generators and relations for the algebra  $A$ , that is  $A \cong k\langle x, y \rangle / I$ , where  $I$  is the ideal of  $k\langle x, y \rangle$ , generated by:

$\{xy + yx, x^3 - \alpha xy^2 - \beta x^2y^2, y^3 - \gamma x^2y - \delta x^2y^2, x^3y, xy^3\} \cup \{u_1u_2u_3u_4u_5, u_i \in \{x, y\}, 1 \leq i \leq 5\}, \alpha, \gamma \in \{0, 1\}, \beta, \delta \in k$ .

*Proof.* By Lemma 5.4, there is a basis  $\{x + J_2, y + J_2\}$  of  $J_1/J_2$  with  $xy + yx = 0$ . Since  $x^3, y^3 \in J_3 \subseteq Z$  by Lemma 3.3, the relation  $xy + yx = 0$  forces  $x^3y = -yx^3$  and  $xy^3 = -y^3x$ . Thus,  $x^3y = xy^3 = 0$ . As noted in the proof of Lemma 5.4,  $\{x^2y + J_4, xy^2 + J_4\}$  is a basis of  $J_3/J_4$  and either  $\{x^3y\}$  or  $\{x^2y^2\}$  spans  $J_4$ . But,  $x^3y = 0$ . Thus,  $x^2y^2$  is a basis of  $J_4$ . Write

$$x^3 = \lambda x^2y + \alpha xy^2 + \beta x^2y^2.$$

Multiplying on the right with  $y$  yields

$$0 = \lambda x^2 y^2$$

hence  $\lambda = 0$ . Similarly,

$$y^3 = \gamma x^2 y + \delta x^2 y^2.$$

If  $\alpha = 0$ , set  $y' = y$ , and if  $\alpha \neq 0$ , set  $y' = \sqrt{\alpha^{-1}}y$ . If  $\gamma = 0$ , set  $x' = x$ , and if  $\gamma \neq 0$ , set  $x' = \sqrt{\gamma^{-1}}x$ . Then replacing  $\{x, y\}$  with  $\{x', y'\}$  gives  $\alpha, \gamma \in \{0, 1\}$ . Thus (i)-(iv) are satisfied. The final assertion follows from the fact that any algebra satisfying (i)-(iii) has dimension at most 9.  $\square$

For  $A$  satisfying Hypothesis 5.3, we will require only partial structure results.

**Lemma 5.6.** *Suppose that  $A$  satisfies hypothesis 5.3. Then,*

- (i)  $\{x^2, xy, yx\}$  is a basis of  $J_2/J_3$ .
- (ii)  $yx y \notin J_4$ .
- (iii)  $\{xyxy\}$  is a basis of  $J_4$ .
- (iv)  $\{xyx + J_4, yxy + J_4\}$  is a basis of  $J_3/J_4$ .

*Proof.* (i) is immediate from the first part of the proof of Lemma 5.1. By Lemma 3.4,  $\{[xy, x], [xy, y]\}$  is a basis of  $J_3/J_4$ . But  $[xy, y] = yxy - y^2x \equiv yxy \pmod{J_4}$ . This proves (ii). Suppose that  $xyxy = 0$ . Since  $yxy \in Z$ , this also means that  $yx yx = 0$ . Since  $y^2 \in J_3$ ,  $yyxy = 0 = yxyy$ . Thus  $yxy \in \text{Soc}(A) = J_4$ , contradicting (ii). Thus,  $xyxy \neq 0$ , proving (iv). Since  $\dim_k(J_3/J_4) = 2$ , by (ii), in order to prove (iv) it suffices to show that  $xyx + J_4 \neq \lambda yxy + J_4$  for  $\lambda \in k$ . If this were the case then multiplying on both sides on the right with  $y$  would give  $xyxy = \lambda yxyy = 0$  and this would contradict (ii) as  $yx yx = xyxy$ .  $\square$

**Lemma 5.7.** *Suppose that  $A$  satisfies Hypothesis 5.3. Then,  $x$  and  $y$  can be chosen such that*

- (i)  $x^2 y \in J_4$ .
- (ii)  $x^3 y = 0$ .
- (iii)  $x^3 \equiv yxy \pmod{J_4}$ .

*Proof.* Let  $\{x + J_2, y + J_2\}$  be a basis of  $J_1/J_2$  with  $y^2 \in J_3$ . By Lemma 5.6, we may write

$$x^2 y \equiv \alpha xyx + \beta yxy \pmod{J_4}.$$

Then,

$$0 = x^2 y^2 = \alpha xyxy.$$

Hence  $\alpha = 0$ . Now

$$(x - \beta y)^2 y \equiv x^2 y - \beta yxy \equiv 0 \pmod{J_4}$$

so replacing  $x$  by  $x' = x - \beta y$  we may assume that (i) holds. (ii) is immediate from (i).

By Lemma 5.6,

$$x^3 \equiv \alpha xyx + \beta yxy \pmod{J_4}.$$

Multiplying with  $y$  on the right yields  $\alpha = 0$ . Now suppose if possible that  $\beta \neq 0$ . Then multiplying with  $x$  yields  $x^4 = 0$ . By (ii)

$$x^2(xy + yx) = 2x^3 y = 0.$$

Since  $\{xy + yx, x^2, xyx, yxy, xyxy\}$  is a basis of  $Z_1 = J(Z)$  and since  $\{xyx, yxy, xyxy\} \subseteq \text{Soc}(Z(A))$ , it follows that  $x^2 \in \text{Soc}(Z(A))$ , and hence that  $\dim_k(\text{Soc}(Z(A))) \geq 4$ , a contradiction. So  $\beta \neq 0$ . Replacing  $x$  by  $x' := \sqrt{\beta^{-1}}x$  yields a basis satisfying (i), (ii), (iii).  $\square$

## 6. OUTER AUTOMORPHISM GROUPS

In this section we will keep to the notation introduced for outer automorphism groups in Section 2.

**Proposition 6.1.** *Suppose that  $A$  satisfies Hypothesis 4.1, and let  $\alpha, \beta$  be as in Proposition 4.5.*

(i) *If either  $\alpha$  or  $\beta$  is non-zero, then  $\text{Out}^0(A)/R_u(\text{Out}^0(A))$  is contained in a one-dimensional torus.*

(ii) *If  $\alpha = \beta = 0$ , then  $\text{Out}^0(A)/R_u(\text{Out}^0(A))$  is a two-dimensional torus.*

*Proof.* Let  $\{x + J_2, y + J_2, z + J_2\}$  be a basis of  $J_1/J_2$  as in Proposition 4.5. Let  $\mathcal{B}$  be the ordered basis  $\{x, y, z, xy, yx, xyx, yxy, xyxy\}$  of  $J_1$  and let  $\varphi : A \rightarrow A$  be a  $k$ -linear map. For each  $u \in \mathcal{B}$ , write  $\varphi(u) = \sum_{v \in \mathcal{B}} \lambda_{v,u} v$ . Suppose that  $\varphi \in \text{Aut}(A)$ . Then,

$$0 \equiv \varphi(x)^2 \equiv \lambda_{x,x} \lambda_{y,x} (xy + yx) \pmod{J_3},$$

$$0 \equiv \varphi(y)^2 \equiv \lambda_{x,y} \lambda_{y,y} (xy + yx) \pmod{J_3},$$

$$\lambda_{x,x} \lambda_{y,x} = \lambda_{x,y} \lambda_{y,y} = 0,$$

that is either  $\lambda_{x,y} = \lambda_{y,x} = 0$  or  $\lambda_{x,x} = \lambda_{y,y} = 0$ . Identifying  $GL(J_1/(J_2 + Z_1))$  with  $GL_2(k)$  through the ordered basis  $\{x + J_2 + Z_1, y + J_2 + Z_1\}$  it follows that the connected component of  $f_{J_1, J_2 + Z_1}(\text{Aut}(A))^0$  is contained in the subgroup of diagonal matrices. Hence, if  $\varphi \in \text{Aut}^0(A)$  then

$$(3) \quad \lambda_{x,y} = \lambda_{y,x} = 0.$$

We will assume from now on that this is the case.

Since  $\varphi(z) \in Z(A)$ ,

$$(4) \quad \lambda_{x,z} = \lambda_{y,z} = 0$$

So,

$$0 \equiv \varphi(z)\varphi(x) \equiv \lambda_{xy,z} \lambda_{x,x} xyx \pmod{J_4}, \text{ and } 0 \equiv \varphi(z)\varphi(y) \equiv \lambda_{yx,z} \lambda_{y,y} yxy \pmod{J_4},$$

$$(5) \quad \lambda_{xy,z} = \lambda_{yx,z} = 0.$$

So,

$$\lambda_{x,x}^2 \lambda_{y,y}^2 xyxy = \varphi(x)\varphi(y)\varphi(x)\varphi(y) = \varphi(z)^2 = \lambda_{z,z}^2 xyxy,$$

yielding

$$\lambda_{z,z}^2 = \lambda_{x,x}^2 \lambda_{y,y}^2.$$

Thus, if  $\varphi \in \text{Aut}^0(A)$  then

$$(6) \quad \lambda_{z,z} = \lambda_{x,x} \lambda_{y,y}.$$

Identify  $GL(J_1/J_2)$  with  $GL_3(k)$  via the ordered basis  $\{x + J_1, y + J_1, z + J_1\}$  of  $J_1/J_2$ . By Equations 3 and 6,  $f_1(\text{Aut}^0(A))$  is contained in the subgroup of lower triangular matrices

$(a_{ij})$  satisfying  $a_{33} = a_{11}a_{22}$ ,  $a_{2,1} = 0$ . So  $R_u(f_1(\text{Aut}^0(A)))$  is the subgroup of  $f_1(\text{Aut}^0(A))$  consisting of matrices for which  $a_{ii} = 1$ ,  $1 \leq i \leq 3$ .

(i) Suppose first that  $\beta \neq 0$ . Comparing the coefficient of  $xyx$  on both sides of the relation  $\varphi(x)^2 = \alpha\varphi(x)\varphi(y)\varphi(x) + \beta\varphi(y)\varphi(x)\varphi(y)$  yields

$$\lambda_{x,x}^2\beta = \lambda_{x,x}\lambda_{y,y}^2\beta,$$

hence,  $\lambda_{x,x} = \lambda_{y,y}^2$ . It follows from the above discussion that  $f_1(\text{Aut}^0(A))/R_u((\text{Aut}^0(A)))$  is isomorphic to a subgroup of the group of diagonal matrices of  $GL_3(k)$  satisfying  $a_{11} = a_{2,2}$  and  $a_{3,3} = a_{2,2}^3$ . So, the result follows by Proposition 2.4. Note that  $f_1(\text{Aut}^0(A)) = f_1(\text{Aut}(A))^0$  (see [3, Proposition 7.4B]).

Similarly, if  $\alpha \neq 0$ , then comparing the coefficient of  $xyx$  in the relation  $\varphi(y)^2 = \alpha\varphi(x)\varphi(y)\varphi(x) + \beta\varphi(y)\varphi(x)\varphi(y)$  yields that  $f_1(\text{Aut}^0(A))/R_u((\text{Aut}^0(A)))$  is isomorphic to a subgroup of the group of diagonal matrices of  $GL_3(k)$  satisfying  $a_{22} = a_{11}$  and  $a_{3,3} = a_{11}^3$ .

(ii) Suppose from now on that  $\alpha = \beta = 0$ . Then,

$$0 = \varphi(x)^2 = \lambda_{x,x}(\lambda_{xy,x} + \lambda_{yx,x})xyx + (2\lambda_{x,x}\lambda_{yxy,x} + \lambda_{xy,x}^2 + \lambda_{yx,x}^2)xyxy,$$

giving

$$(7) \quad \lambda_{yx,x} = -\lambda_{xy,x}, \quad \lambda_{yxy,x} = -\lambda_{xy,x}^2\lambda_{x,x}^{-1}.$$

Similarly,

$$(8) \quad \lambda_{xy,y} = -\lambda_{yx,y}, \quad \lambda_{xyx,y} = -\lambda_{yx,y}^2\lambda_{y,y}^{-1}.$$

Then,

$$0 = \varphi(z)\varphi(x) = (\lambda_{z,z}\lambda_{z,x} + \lambda_{x,x}\lambda_{yxy,z})yxyx,$$

and

$$0 = \varphi(z)\varphi(y) = (\lambda_{z,z}\lambda_{z,y} + \lambda_{y,y}\lambda_{xyx,z})yxyx,$$

which gives

$$(9) \quad \lambda_{yxy,z} = -\lambda_{z,z}\lambda_{z,x}\lambda_{x,x}^{-1}, \quad \lambda_{xyx,z} = -\lambda_{z,z}\lambda_{z,y}\lambda_{y,y}^{-1}$$

Conversely, it is easy to check that any element of  $GL(J)$  satisfying the Equations 3, 4, 5, 6, 7, 8 and 9 and such that the image under  $\varphi$  of elements of  $\mathcal{B} \cap J_2$  is the multiplicative extension of the images of  $x, y$  and  $z$  of  $\varphi$  is in  $\text{Aut}^0(A)$ . So, identifying as in (i),  $GL(J_1/J_2)$  with  $GL_3(k)$  via the ordered basis  $\{x + J_1, y + J_1, z + J_1\}$  of  $J_1/J_2$ ,  $f_1(\text{Aut}^0(A))$  is the subgroup of lower triangular matrices  $(a_{ij})$  satisfying  $a_{33} = a_{11}a_{22}$ ,  $a_{2,1} = 0$ ,  $R_u(f_1(\text{Aut}^0(A)))$  is the subgroup of  $f_1(\text{Aut}^0(A))$  consisting of matrices which satisfy additionally that  $a_{1,1} = a_{2,2} = a_{3,3} = 1$  and  $f_1(\text{Aut}^0(A))/R_u(f_1(\text{Aut}^0(A)))$  is isomorphic to the subgroup of diagonal matrices satisfying  $a_{3,3} = a_{1,1}a_{2,2}$ . So, (ii) follows by Proposition 2.4.  $\square$

For the next result, note that the centre of an affine algebraic group is a closed subgroup of the group, and hence it makes sense to speak of the dimension of the centre as an algebraic variety.

**Proposition 6.2.** *Suppose that  $A$  satisfies Hypothesis 4.1 and that the elements  $\alpha, \beta$  Proposition 4.5 are both equal to 0. Then*

$$\dim(Z(R_u(\text{Out}^0(A)))) = 3.$$

*Proof.* Let  $\mathbf{U} = R_u(\text{Out}^0(A))$ , let  $\{x+J_2, y+J_2, z+J_2\}$  be a basis of  $J_1/J_2$  as in Proposition 4.5 and let  $\mathcal{B}$  be the ordered basis  $\{x, y, z, xy+yx, xy-yx, xyx, yxy, xyx, xyxy\}$  of  $J$  (note the difference with the basis used in Proposition 6.1). Identify  $\text{Aut}^0(A)$  with a subgroup of  $GL_8(k)$  through the basis  $\mathcal{B}$ . Let  $B_8(k)$  be the subgroup of  $GL_8(k)$  consisting of lower triangular matrices and let  $U_8(k)$  be the subgroup of  $B_8(k)$  of strictly unitriangular matrices. Let  $\varphi \in GL_8(k)$  be such that the image under  $\varphi$  of elements of  $\mathcal{B} \cap J_2$  is the multiplicative and linear extension of the images of  $x, y$  and  $z$  under  $\varphi$ . By the proof of Proposition 6.1,  $\varphi \in \text{Aut}^0(A)$  if and only if

$$\begin{aligned} \lambda_{x,y} &= \lambda_{y,x} = \lambda_{xy,z} = \lambda_{yx,z} = 0; \\ \lambda_{xy+yx,x} &= \lambda_{xy+yx,y} = \lambda_{xy+yx,z} = \lambda_{xy-yx,z} = 0; \\ \lambda_{yxy,x} &= -\lambda_{xy-yx,x}^2 \lambda_{x,x}^{-1}, \lambda_{xyx,y} = -\lambda_{xy-yx,y}^2 \lambda_{y,y}^{-1}; \\ \lambda_{yxy,z} &= -\lambda_{z,z} \lambda_{z,x} \lambda_{x,x}^{-1}, \lambda_{xyx,z} = -\lambda_{z,z} \lambda_{z,y} \lambda_{y,y}^{-1}. \end{aligned}$$

In particular,  $\text{Aut}^0(A)$  is contained in  $B_8(k)$  and  $R_u(\text{Aut}^0(A)) = \text{Aut}^0(A) \cap U_8(k)$ . Thus  $R_u(\text{Aut}^0(A))$  is the following closed subgroup of  $U_8(k)$ :

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & e & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ b & f & 0 & 0 & 1 & 0 & 0 & 0 \\ c & -f^2 & -e & 0 & -2f & 1 & 0 & 0 \\ -b^2 & g & -a & 0 & -2b & 0 & 1 & 0 \\ d & h & i & 2(g+ae+2bf+c) & 0 & 0 & 0 & 1 \end{pmatrix} : a, b, c, d, e, f, g, h, i \in k \right\}$$

We now calculate  $\text{Inn}(A)$ . Let  $u \in J_1$ . So by Lemma 2.6,  $\varphi_u(v) = v + [v, u] + [vu, u]$  for all  $v \in V$ . By Lemma 3.3,  $Z \cap J_1$  is an ideal of  $A$ , hence we may assume that  $u = \epsilon_1 x + \epsilon_2 y + \epsilon_3(xy - yx)$ ,  $\epsilon_i \in k$ ,  $1 \leq i \leq 3$ . By Lemma 3.3 one easily calculates

$$\begin{aligned} \varphi_u(x) &= x + \epsilon_2(xy - yx) + (\epsilon_1\epsilon_2 - 2\epsilon_3)xyx - \epsilon_2^2yxy; \\ \varphi_u(y) &= y - \epsilon_1(xy - yx) - \epsilon_1^2xyx + (\epsilon_1\epsilon_2 + 2\epsilon_3)yxy; \\ \varphi_u(z) &= 1. \end{aligned}$$

Thus  $\text{Inn}(A)$  consist of matrices above such that  $a = e = d = h = i = 0$  and  $c + g = -2bf$ .

For each  $\varphi \in \text{Aut}^0(A)$ , let  $\varphi^0 := \varphi - \text{Id}$ . Then from the above description of  $\text{Aut}^0(A)$ , we have that for all  $\varphi \in \text{Aut}^0(A)$ ,  $\{xyx, yxy, xyxy\} \subset \text{Ker}(\varphi^0)$ ,  $\varphi^0(xy - yx)$ ,  $\varphi^0(xy + yx)$  and  $\varphi^0(z)$  are contained in the  $k$ -span of  $\{xyx, yxy, xyxy\}$  and  $\varphi^0(x)$  and  $\varphi^0(y)$  are contained in the  $k$ -span of  $\{z, xy - yx, xy + yx, xyx, yxy, xyxy\}$ . Hence,  $\varphi^0 \tau^0 \psi^0 = 0$  for all  $\varphi, \tau, \psi \in \text{Aut}^0(A)$ . It follows that for all  $\varphi \in \text{Aut}^0(A)$ ,  $\varphi^{-1} = \text{Id} + \varphi^0 + \varphi^{0^2}$  and for all  $\varphi, \tau \in \text{Aut}^0(A)$

$$\varphi \tau \varphi^{-1} \tau^{-1} = \text{Id} + \varphi^0 \tau^0 - \tau^0 \varphi^0.$$

Further, since  $\{z, xy - yx, xy + yx, xyx, yxy, xyxy\}$  is contained in the kernel of  $\varphi^0 \tau^0$ ,

$$\varphi^0 \tau^0(u) - \tau^0 \varphi^0(u) = 0$$



for all  $u \in \mathbb{B}$  different from  $x, y$ . Let  $\varphi$  be the matrix displayed above and let  $\tau$  be the matrix with  $a$  replaced by  $a'$  etc. Then,

$$\begin{aligned}\varphi^\circ \tau^\circ(x) - \tau^\circ \varphi^\circ(x) &= (ae' - a'e + 2bf' - 2b'f)xyx + (a'i - ai')xyxy, \\ \varphi^\circ \tau^\circ(y) - \tau^\circ \varphi^\circ(y) &= (ea' - e'a + 2fb' - 2f'b)xyy + (e'i - ei')xyxy.\end{aligned}$$

Thus, for any  $\varphi \in \text{Aut}^0(A)$ ,  $\varphi\tau\varphi^{-1}\tau^{-1} \in \text{Inn}(A)$  for all  $\tau \in \text{Aut}^0(A)$  if and only if

$$a = e = i = 0.$$

Let  $\hat{\mathbf{Z}}$  be the inverse image in  $R_u(\text{Aut}^0(A))$  of  $Z(\mathbf{U})$ . There is an obvious injective morphism of varieties  $k^6 \rightarrow GL_8(k)$  whose image is  $\hat{\mathbf{Z}}$  and  $\hat{\mathbf{Z}}$  is a closed subgroup of  $\text{Aut}^0(A)$  and hence of  $GL_8(k)$ . Thus,  $\hat{\mathbf{Z}}$  has dimension 6 (see [3, Corollary 4.3]). Similarly  $\text{Inn}(A)$  has dimension 3. So, it follows that  $Z(\mathbf{U})$  has dimension 3 (see [3, Proposition 7.4B]). □

Note that if  $A$  is as in Proposition 5.5 with  $\alpha = \beta = \gamma = \delta = 0$ , then  $A \cong B$ .

**Proposition 6.3.** *Suppose that  $A$  satisfies Hypothesis 5.2 and that  $\alpha, \beta, \gamma, \delta$  are as in Proposition 5.5.*

(i) *If any one of  $\alpha, \beta, \gamma$  or  $\delta$  is non-zero, then  $\text{Out}^0(A)/R_u(\text{Out}^0(A))$  is contained in a 1-dimensional torus.*

(ii) *If  $\alpha = \beta = \gamma = \delta = 0$ , then  $\text{Out}^0(A)/R_u(\text{Out}^0(A))$  is a two-dimensional torus.*

*Proof.* Let  $\{x + J_2, y + J_2\}$  be a basis of  $J_1/J_2$  as in Proposition 5.5. Let  $\mathcal{B}$  be the ordered basis  $\{x, y, x^2, y^2, xy, yx, yxy, xyxy\}$  of  $J_1$ . Let  $\varphi \in \text{Aut}(A)$  and for  $u \in \mathcal{B}$  write  $\varphi(u) = \sum_{v \in \mathcal{B}} \lambda_{v,u} v$ . Then,

$$0 \equiv \varphi(x)\varphi(y) + \varphi(y)\varphi(x) \equiv \lambda_{x,x}\lambda_{x,y}x^2 + \lambda_{y,x}\lambda_{y,y}y^2 \pmod{J_3},$$

So,

$$\lambda_{x,x}\lambda_{y,x} = \lambda_{x,y}\lambda_{y,y} = 0,$$

that is either  $\lambda_{x,y} = \lambda_{y,x} = 0$  or  $\lambda_{x,x} = \lambda_{y,y} = 0$ . Identifying  $GL(J_1/J_2)$  with  $GL_2(k)$  through the ordered basis  $\{x + J_2, y + J_2\}$  it follows that  $f_1(\text{Aut}(A))^0$  is contained in the subgroup of diagonal matrices. Hence, if  $\varphi \in \text{Aut}^0(A)$  then

$$(10) \quad \lambda_{x,y} = \lambda_{y,x} = 0.$$

We will assume from now on that this is the case.

(i) Now suppose that  $\alpha \neq 0$ . Comparing the coefficient of  $xy^2$  in the relation  $\varphi(x)^3 = \alpha\varphi(x)\varphi(y)^2 + \beta\varphi(x)^2\varphi(y)^2$  gives

$$\lambda_{x,x}^3\alpha = \lambda_{x,x}\lambda_{y,y}^2\alpha,$$

and hence  $\lambda_{x,x}^2 = \lambda_{y,y}^2$ . Thus, identifying  $f_1(\text{Aut}(A))^0 = f_1(\text{Aut}^0(A))$  with its image in  $GL_2(k)$  as above,  $f_1(\text{Aut}(A))^0$  is contained in the subgroup of diagonal matrices  $(a_{ij})$  satisfying  $a_{1,1} = a_{2,2}$ . This proves (i) if  $\alpha \neq 0$  and similarly if  $\gamma \neq 0$ . Suppose now that  $\alpha = \gamma = 0$  and  $\beta \neq 0$ . Comparing the coefficient of  $x^2y^2$  in the relation  $\varphi(x)^3 = \beta\varphi(x)^2\varphi(y)^2$  gives

$$\lambda_{x,x}^3\beta + 3\lambda_{x,x}^2\lambda_{y^2,x} = \beta\lambda_{x,x}^2\lambda_{y,y}^2,$$

and hence (since  $k$  is of characteristic 3),

$$\lambda_{x,x} = \lambda_{y,y}^2.$$

Thus  $f_1(\text{Aut}(A))^0$  is contained in the subgroup of diagonal matrices  $(a_{ij})$  satisfying  $a_{1,1} = a_{2,2}^2$ . This proves (i) if  $\alpha = \gamma = 0$ ,  $\beta \neq 0$ . The proof for the case that  $\alpha = \gamma = 0$ ,  $\delta \neq 0$  is similar.

(ii) Suppose that  $\alpha = \beta = \gamma = \delta = 0$ . The coefficient of  $x^2y$  in  $\varphi(x)\varphi(y) + \varphi(y)\varphi(x)$  is  $2\lambda_{x,x}\lambda_{y^2,y} = 0$  and the coefficient of  $xy^2$  in  $\varphi(x)\varphi(y) + \varphi(y)\varphi(x)$  is  $2\lambda_{y,y}\lambda_{x^2,x}$ , hence

$$(11) \quad \lambda_{x,x^2} = \lambda_{y,y^2} = 0.$$

Now the coefficient of  $x^2y^2$  in  $\varphi(x)\varphi(y) + \varphi(y)\varphi(x)$  is

$$2\lambda_{x,x}\lambda_{xy^2,y} - 2\lambda_{xy,x}\lambda_{xy,y} + 2\lambda_{y,y}\lambda_{xy^2,y} + 2\lambda_{y^2,x}\lambda_{x^2,y},$$

hence

$$(12) \quad \lambda_{xy^2,y} = (\lambda_{xy,x}\lambda_{xy,y} - \lambda_{x,x}\lambda_{xy^2,y} - \lambda_{x^2,y}\lambda_{y^2,x})\lambda_{y,y}^{-1}$$

Conversely, any element of  $GL(J)$  satisfying the Equations 10, 11 and 12, and such that the image under  $\varphi$  of elements of  $\mathcal{B} \cap J_2$  is the obvious multiplicative extension of the images of  $x, y$  and  $z$  of  $\varphi$  is in  $\text{Aut}^0(A)$ . So, identifying as in (i),  $GL(J_1/J_2)$  with  $GL_3(k)$  via the ordered basis  $\{x + J_1, y + J_1\}$  of  $J_1/J_2$ ,  $f_1(\text{Aut}^0(A))$  is the subgroup of diagonal matrices.  $\square$

The next result gives the dimension of  $Z(R_u(\text{Out}^0(B)))$ .

**Proposition 6.4.** *Suppose that  $A$  satisfies Hypothesis 5.2 and that the elements  $\alpha, \beta, \gamma, \delta$  of Proposition 5.5 are all equal to 0. Then  $\dim(Z(R_u(\text{Out}^0(A)))) = 2$ .*

*Proof.* Let  $\{x + J_2, y + J_2\}$  be a basis of  $J_1/J_2$  as in Proposition 5.5. Let  $\mathbf{U} = R_u(\text{Out}^0(A))$  and let  $\mathcal{B}$  be the ordered basis  $\{x, y, x^2, y^2, xy, x^2y, xy^2, x^2y^2\}$  of  $J$  and identify  $\text{Aut}^0(A)$  with a subgroup of  $GL_8(k)$  through the basis  $\mathcal{B}$ . As in the Proposition 6.4, let  $B_8(k)$  be the subgroup of  $GL_8(k)$  consisting of lower triangular matrices and let  $U_8(k)$  be the subgroup of  $B_8(k)$  of strictly unitriangular matrices. By Equations, 10, 11 and 12,  $\text{Aut}^0(A)$  is contained in  $B_8(k)$  and hence by Lemma 2.5(i),  $R_u(\text{Aut}^0(A)) = \text{Aut}^0(A) \cap U_8(k)$ . Further,  $R_u(\text{Aut}^0(A))$  is the following subgroup of  $U_8(k)$ :

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 1 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ b & f & 0 & 0 & 1 & 0 & 0 & 0 \\ i & g & 0 & -e & f & 1 & 0 & 0 \\ c & -bf - ae - i & -a & 0 & b & 0 & 1 & 0 \\ d & h & -c - b^2 & -g - f^2 & 0 & 0 & 0 & 1 \end{pmatrix} : a, b, c, d, ef, g, h, i \in k \right\}$$

We now calculate  $\text{Inn}(A)$ . Let  $u \in J_1$ . By Lemma 2.6,  $\varphi_u(v) = v + [v, u] + [vu, u]$  for all  $v \in V$ . By Lemma 3.3,  $Z \cap J_1$  is an ideal of  $A$ , hence we may assume that  $u = \epsilon_1x + \epsilon_2y + \epsilon_3xy$ ,  $\epsilon_i \in k$ ,  $1 \leq i \leq 3$ . One easily calculates

$$\begin{aligned} \varphi_u(x) &= x + 2\epsilon_2xy + (2\epsilon_3 - 2\epsilon_1\epsilon_2)x^2y + 2\epsilon_2^2xy^2; \\ \varphi_u(y) &= y - 2\epsilon_1xy + 2\epsilon_1^2x^2y - 2(\epsilon_1\epsilon_2 + \epsilon_3)xy^2. \end{aligned}$$

Thus  $\text{Inn}(A)$  consist of matrices above such that  $a = e = d = h = 0$  and  $c = -b^2$ ,  $g = -f^2$ .

For each  $\varphi \in \text{Aut}^0(A)$ , let  $\varphi^0 := \varphi - \text{Id}$ . Then from the above description of  $\text{Aut}^0(A)$ , we have that for all  $\varphi \in \text{Aut}^0(A)$ ,  $\{x^2y, xy^2, x^2y^2\} \subset \text{Ker}(\varphi^0)$ ,  $\varphi^0(xy)$ ,  $\varphi^0(x^2)$  and  $\varphi^0(y^2)$  are contained in the  $k$ -span of  $\{x^y, xy^2, x^2y^2\}$  and  $\varphi^0(x)$  and  $\varphi^0(y)$  are contained in the  $k$ -span of  $\{x^2, y^2, xy, x^2y, xy^2, x^2y^2\}$ . Hence,  $\varphi^\circ \tau^\circ \psi^\circ = 0$  for all  $\varphi, \tau, \psi \in \text{Aut}^0(A)$ . It follows that for all  $\varphi \in \text{Aut}^0(A)$ ,  $\varphi^{-1} = \text{Id} + \varphi^\circ + \varphi^{\circ 2}$  and for all  $\varphi, \tau \in \text{Aut}^0(A)$

$$\varphi \tau \varphi^{-1} \tau^{-1} = \text{Id} + \varphi^\circ \tau^\circ - \tau^\circ \varphi^\circ.$$

Further, since  $\{z, xy - yx, xy + yx, xyx, yxy, xyxy\}$  is contained in the kernel of  $\varphi^\circ \tau^\circ$ ,

$$\varphi^\circ \tau^\circ(u) - \tau^\circ \varphi^\circ(u) = 0$$

for all  $u \in \mathbb{B}$  different from  $x, y$ . Let  $\varphi$  be the matrix displayed above and let  $\tau$  be the matrix with  $a$  replaced by  $a'$  etc. Then,

$$\varphi^\circ \tau^\circ(x) - \tau^\circ \varphi^\circ(x) = (ae' - a'e + b'f - bf')x^2y + (a(g' + f'^2) - a'(g + f^2))x^2y^2,$$

$$\varphi^\circ \tau^\circ(y) - \tau^\circ \varphi^\circ(y) = (ea' - e'a + fb' - f'b)xy^2 + (e(c' + b'^2) - e'(c + b^2))x^2y^2.$$

Thus, for any  $\varphi \in \text{Aut}^0(A)$ ,  $\varphi \tau \varphi^{-1} \tau^{-1} \in \text{Inn}(A)$  for all  $\tau \in \text{Aut}^0(A)$  if and only if

$$a = e = 0 \text{ and } c = -b^2, g = -f^2.$$

Arguing as for Proposition 6.2, the inverse image in  $R_u(\text{Aut}^0(A))$  of  $Z(\mathbf{U})$  has dimension 5,  $\text{Inn}(A)$  has dimension 3, and hence  $Z(\mathbf{U})$  has dimension 2, as claimed.  $\square$

**Proposition 6.5.** *Suppose that  $A$  satisfies Hypothesis 5.3. Then  $\text{Out}^0(A)/R_u(\text{Out}^0(A))$  is contained in a one-dimensional torus.*

*Proof.* Let  $\{x + J_2, y + J_2\}$  be a basis of  $J_1/J_2$  satisfying the properties of Lemmas 5.6 and 5.7. Let  $\mathcal{B}$  be the ordered basis  $\{x, y, x^2, xy, yx, xyx, yxy, xyxy\}$  of  $J_1$  and let  $\varphi \in \text{Aut}(A)$  be as before. The fact that  $\varphi(y)^2 \equiv 0 \pmod{J_3}$  implies that  $\lambda_{x,y} = 0$  and we will assume that this is the case. Now consider the equation

$$\varphi(x)^3 \equiv \varphi(y)\varphi(x)\varphi(y) \pmod{J_4}$$

$$\lambda_{x,x}^3 x^3 + \lambda_{x,x}^2 \lambda_{y,x} xyx + 2\lambda_{x,x}^2 \lambda_{y,x} x^2y + \lambda_{x,x} \lambda_{y,x}^2 yxy \equiv \lambda_{y,y}^2 y(\lambda_{x,x}x + \lambda_{y,x}y)y \pmod{J_4},$$

$$\lambda_{x,x}^2 \lambda_{y,x} xyx + (\lambda_{x,x}^3 + \lambda_{x,x} \lambda_{y,x}^2) yxy \equiv \lambda_{y,y}^2 \lambda_{x,x} yxy \pmod{J_4},$$

$$\lambda_{x,x}^2 \lambda_{y,x} = 0, \quad \lambda_{x,x}^3 + \lambda_{x,x} \lambda_{y,x}^2 = \lambda_{y,y}^2 \lambda_{x,x},$$

$$\lambda_{y,x} = 0, \quad \lambda_{x,x}^3 = \lambda_{y,y}^2 \lambda_{x,x},$$

$$\lambda_{y,x} = 0, \quad \lambda_{y,y} = \pm \lambda_{x,x}.$$

The result is follows from these calculations and Proposition 2.4.  $\square$

## 7. PROOF OF THEOREM 1.2 AND THEOREM 1.1

**Proof of Theorem 1.2.** There is a stable equivalence of Morita type between  $A$  and  $B$ , by [16, Théorème 4.15],  $\text{Out}^0(A)$  and  $\text{Out}^0(B)$  are isomorphic as algebraic groups. In particular,  $\text{Out}^0(A)/R_u(\text{Out}^0(A)) \cong \text{Out}^0(B)/R_u(\text{Out}^0(B))$  and  $R_u(\text{Out}^0(A)) \cong R_u(\text{Out}^0(B))$ . By Proposition 6.3, and the remark preceding it,  $\text{Out}^0(B)/R_u(\text{Out}^0(B))$  is a 2-dimensional torus, and by Proposition 6.4,  $\dim(Z(R_u(\text{Out}^0(A)))) = 2$ . Hence, it follows from Propositions 6.1, 6.2, 6.3, and 6.5 that if  $A$  satisfies one of the hypotheses 4.1, 5.2 or 5.3, then  $A \cong B$ . The result follows as by Proposition 3.5 and Lemma 5.1,  $A$  does satisfy one of the hypothesis 4.1, 5.2 or 5.3.  $\square$

**Proof of Theorem 1.1.** By the structure theory of blocks with normal defect groups [7],  $C$  is Morita equivalent to a twisted group algebra  $k_\alpha P \rtimes E$ , where  $E$  is a  $p'$ -subgroup of the automorphism group of  $P$  and  $\alpha$  is an element of  $H^2(E, k^\times)$ . Since  $A$  is not nilpotent and  $C$  has up to isomorphism only one simple module, it is well known (see, for instance, [2, Theorem 1.1]), that  $E$  is a Klein-4-group and that  $C$  is Morita equivalent to the algebra  $B = k\langle X, Y \rangle / \langle X^3, Y^3, XY + YX \rangle$ .

By Rouquier's work [16, 6.3] (see also [10, Theorem A.2]) there is a stable equivalence of Morita type between  $A$  and  $C$ . By [12] (see also [4]), the blocks  $A$  and  $C$  are perfectly isometric. Hence by [1], the centers of  $A$  and  $C$  are isomorphic as  $k$ -algebras and up to isomorphism  $A$  has one simple module. Consequently, the dimension of a basic algebra of  $A$  is equal to 9. The result follows from Theorem 1.2 applied to  $B$  and a basic algebra of  $A$ .  $\square$

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF ABERDEEN, FRASER NOBLE BUILDING, KING'S COLLEGE, ABERDEEN AB24 3UE, U.K.